

2007

MATHEMATICS

THIRD PAPER

(Algebra)

Full Marks: 80

Time: 3 hours

The figures in the margin indicate full marks for the questions

PART-A (Objective-type Questions)

(Marks: 32)

1. Fill in the blanks: 4
- (a) The cyclic subgroup of Z_{24} generated by 18/ has order_____.
- (b) $Z_3 \times Z_4$ is of order_____.
- (c) The element (4, 2) of $Z_{12} \times Z_8$ has order _____.
- (d) The Klein 4-group is isomorphic to_____.
2. Classify the group
 $Z_4 \times Z_6 / \langle (2, 3) \rangle$
Where $(2, 3) \in Z_4 \times Z_6$ and $\langle (2, 3) \rangle$ is the subgroup of $Z_4 \times Z_6$ generated by (2, 3). 4
3. Give an example of a subnormal series which is not a normal series. 4
4. Justify the statement “Z has no composition series”. [Z is the additive group of integers.] 4
5. Find the value of
- (i) $(x+1)^2$
- (ii) $(x+1)+(x+1)$
in $Z_2[x]$. 2+2
6. Write the ideals of Z. Write the maximal ideal of Z. 4
7. The polynomial $p(x) = x^2+x+1$ in $Z_2[x]$ is irreducible over Z_2 . Justify 4
8. Mark whether the following statement is true or not and justify your answer: 4
Q is an extension field of Z_2 (with usual notations).

PART-B (Subjective-type Questions)

(Marks: 48)

Answer any three questions

9. (a) If G is a group and H, K are subgroups of G such that
 $G = H \times K$
Prove that

$$H \cong G/K \text{ and } K \cong G/H \quad 6$$

- b) Let H be a normal subgroup of a group G . If both H and G/H are solvable, then show that G is solvable. 6
- c) If two groups H and K are solvable, then show that $H \times K$ is also solvable. 4
10. (a) Define Euclidean domain with one example. Show that every field is a Euclidean domain. 2+4
- (b) In $\mathbb{Z} / (6)$, show that 2 is a prime element but not irreducible. 6
- (c) State Eisenstein's irreducibility criterion for a polynomial over the field \mathbb{Q} . Use it to show that the following polynomials are irreducible over \mathbb{Q} : 2+2
- (i) $x^4 - 4x + 2$
- (ii) $x^3 - 9x + 15$
11. (a) If L is a finite extension of a field K and K is a finite extension of a field F , then show that L is a finite extension of F . 6
- (b) If a field F has q elements, then show that F is a splitting field of $x^q - x$ over its prime subfield. 6
- (c) Show that $\sqrt{2} + \sqrt[3]{5}$ is algebraic over \mathbb{Q} . 4
12. (a) For any two subspaces W_1 and W_2 of a vector space V_F , show that $W_1 + W_2$ is the subspace of V spanned by $W_1 \cup W_2$. 6
- (b) Let S and T be the linear operators on \mathbb{R}^2 defined by $S(x, y) = (0, x)$ and $T(x, y) = (x, 0)$. Show that
- (i) $TS = 0$
- (ii) $ST \neq 0$
- (iii) $T^2 = T$
- (c) Find the matrix representation of the following linear operator T on \mathbb{R}^3 relative to the usual basis $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$:
 $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by
 $T(x, y, z) = (2x - 3y + 4z, 5x - y + 2z, 4x + 7y)$
13. (a) Suppose A and B are similar matrices. Show that A and B have the same characteristic polynomial. 4
- (b) Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

Is A similar to a diagonal matrix? If so, find one such matrix. 6

- (c) Determine all possible Jordan canonical forms for a linear operator $T: V \rightarrow V$ whose characteristic polynomial is 6

$$\Delta(t) = (t - 2)^3 (t - 5)^2$$