2007

MATHEMATICS

SECOND PAPER

(Topology and Functional Analysis)

Full Marks: 80

Time: 3 hours

The figures in the margin indicate full marks for the questions

PART-A (Objective-type Questions)

(Marks: 32)

Each question (1-16) carries four codes (a), (b), (c) and (d), out of which one is for correct answer. Choose the correct code: **2×16=32**

1. Let X be a set and T_{ic} be the collection of all subsets U of X such that X-U is either countable or is all of X. Then the topology T_{ic} on X is called

a) Cofinite
b) Cocountable
c) Discrete
d) Indiscrete

2. (R, u) is the usual topological space on R. Then the closure of the set of natural numbers (N) is

(a) R
(b) N
(c) N U{0}
(d) None

3. Let A = (a, b), B = [a, b) and C = (a, b]. Then

(a) $A^0 \subset B^0 \subset C^0$

(b) $B^0 \subset A^0 \subset C^0$

(c) $C^0 \subset B^0 \subset A^0$

(d) $A^0 = B^0 = C^0$

4. Let (R, u) be usual topological space and T be relative topology on [0, 1]. Then

(a) (½, 1] is u-open but not T-open
(b) (½, 1] is both u-open and T-open
(c) (½, 1] is neither u-open and nor T-open
(d) (½, 1] is not u-open but T-open

5. Let X = {1, 2, 3, 4, 5} and T = {Φ, {1}, {2, 3} {1, 2, 3}, X} is a topology on X. Let B_1 = {{1, 2}, X}, B_2 = {{2, 3}, X}, B_3 = {{1, 2, 3}}. Then

(a) B_1 is a local base at 1
(b) B_2 is a local base at 3
(c) B_2 is a local base at 2
(d) B_2 is not a local base at 2

6. Let I_x: $(X, T_1) \rightarrow (X, T_2)$ be identity map from topological space (X, T_1) into (X, T_2). Then

(a) $T_1 \subset T_2 \quad if\ I_x$ is an open map

(b) $T_1 \subset T_2 \quad if\ I_x$ is a continuous map

(c) $T_2 \subset T_1 \quad if\ I_x$ is open

(e) None of (a), (b) and (c)

7. If R is equipped with usual topology, then
(a) (0,1) is a compact subset of R
(b) (0,1) U[1,2) is a connected set
(c) {0, 1} is a compact subset of R
(d) [0, 1) is a compact and connected set

8. (a) A Hausdorff space is metrizable
(b) A metric space is a Hausdorff space
(c) Hausdorff property is not hereditary
(d) Hausdorff is not a topological property

9. Continuity of linear function is characterised by
(a) Hahn-Banach theorem
(b) The open mapping theorem
(c) The closed graph theorem
(d) The principle of uniform bounded ness

10. The law of parallelogram is nor satisfied by
(a) l_2
(b) l_1
(c) l_3
(d) l_4

11. Let r_1 be the radius of the closure of the open ball $B(x_0, r)$. Then
(a) $r_1 > r$
(b) $r_1 < r$
(c) $r_1 = r$
(d) None of (a), (b) and (c)

12. In an inner product space if $A \subseteq B$, then

(a) $A^{\perp} = B^{\perp}$

(b) $A^{\perp} \subseteq B^{\perp}$

(c) $B^{\perp} \subseteq A^{\perp}$

(d) None of (a), (b) and (c)

13. Orthonormal set derived from $x_1 = (3, 0, 4)$, $x_2 = (-1, 0, 7)$ and $x_3 = (2, 9, 1)$ is

(a) $\frac{1}{5}(3,0,4), \quad \frac{1}{5}(-4,0,3), (0,1,0)$

(b) $\frac{1}{5}(4,0,3), \frac{1}{5}(-4,0,3), (0,1,0)$

(c) $(1,0,0), \quad \frac{1}{5}(-4,0,3), \frac{1}{\sqrt{85}}(2,9,1)$

(d) $(3,0,4), \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{85}}(2,9,1)$

14. If T^* be adjoint operator of T, then

(a) $(T^*)^{-1} = (T^{-1})^*$

(b) $(T_1 T_2)^* = T_1^* T_2^*$

(c) $(\alpha T)^* = \alpha T^*$

(d) $\langle Sx, x\rangle = \langle x, Tx\rangle \Rightarrow S^* = T^*$

15. In an inner product space if x⊥y, then

(a) $\|x+y\|^2 > \|x\|^2 + \|y\|^2$

(b) $\|x+y\|^2 < \|x\|^2 + \|y\|^2$

(c) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

(d) None of (a), (b) and (c)

16. In inner product space (C^n, ⟨.⟩) $\langle u, v\rangle = u_1\overline{v_1} + u_2\overline{v_2} + ... + u_n\overline{v_n}$ with if u = (i, i, i, …i), then || u || is equal to

(a) n (b) √n (c) n^2 (d) none

PART-B (Subjective-type Question)

(Marks: 48)

Answer any **three** parts of each question (**17-20**): **12×4 = 48**

17. (a) Consider the topology T= {ϕ, {a}, {c, d}, {a, c, d},{b, c, d, e}, X} on the set X = {a, b, c, d, e}. Find the relative topology of T on A = {a, d, e}.

(b) Show that the class $\wp$ = {[a, b)| a, b rational, a<b} is a basis for a topology on

R.

(c) Prove that a mapping f from a topological space X to a topological space Y is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$, for every subset $A \subseteq X$.

(d) Prove that every metric space is first countable.

(e) Prove that every regular T_1 space is a T_2 space.

18. (a) Show that a metric space is compact if and only if it is sequentially compact.

(b) Show that every compact space has the Bolzano - Weierstrass property.

(c) Prove that a set A is connected if and only if A is not union of two non-empty separated sets.

(d) Let A be a connected subset of topological space X. If $A \subset B \subset \overline{A}$ then show that B is also connected.

(e) Prove that every projection $\Pi_i : X \to X_i$ on a product space $X = \prod_i X_i$ is both open and continuous

19. (a) In a normed linear space $(X, \|.\|)$ if A is compact and B is closed, then prove that A+B is closed

(b) Prove that T**:** $l_\infty \to l_\infty$

$$x = (x_1, x_2, x_3, ...) \to T_x = (\frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, ...)$$

is a bounded linear mapping.

(c) Prove that two $\|.\|_1$ *and* $\|.\|_2$ norms on a linear space X are equivalent if and only if there exist two numbers a, b>0 such that

$$a\|x\|_1 \le \|x\|_2 \le b\|x\|_1 \quad \text{for all} \quad x \in X.$$

(d) Show that dual space of c_0 is l_1.

(e) State and prove 'the closed graph theorem'.

20. (a) Show that in an inner product space $(X\langle,\rangle)$, the parallelogram law holds.

b) Prove that a non-empty closed convex set C in a Hilbert space has a unique point of minimal norm.

c) If M is a closed subspace of a Hilbert space H, then prove that $X=M \oplus M^{\perp}$.

d) Let X be a Hilbert space and (e_i) a complete orthonormal sequence. Then for an $x \in X$, prove that

$$x = \sum_{j=1}^{\infty} \langle x, e_j \rangle e_j \quad \text{and} \quad \|x\|^2 = \sum_{j=1}^{\infty} \left|\langle x, e_j \rangle\right|^2$$

(e) If T is normal and μ is scalar, then show that $T-\mu I$ is normal and

$$\|T^* x - \overline{\mu} x\| = \|Tx - \mu x\|$$

for all $x \in X$.