2007

MATHEMATICS

FIRST PAPER

(Real and Complex Analysis)

Full Marks: 80

Time: 3 hours

The figures in the margin indicate full marks for the questions

PART-A (Objective-type Questions)

(Marks: 32)

Each question (1-16) carries four codes (a), (b), (c) and (d), out of which one is for correct answer. Choose the correct code: 2 x 16=32

1. The RS-integral

$$\int_0^3 x^2 d([x]-x)$$

is equal to

a) 2
b) 3
c) 8
d) 5

2. If $\{x/(1+nx^2)\}$ converges uniformly to f on [0,1], then

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is true if

(a) $x \neq 0$
(b) $x>0$
(c) $x<0$
(d) $x=0$

3. If f is of bounded variation on [a, b] and $a \leq c \leq b$, then

(a) V(f, a, b) – V(f, a, c) < V(f, c, b)
(b) V(f, a, b) – V(f, a, c) = V(f, c, b)
(c) V(f, a, b)- V(f, a, c)> V(f, c, b)
(d) V(f, a, b) + V(f, c, b)= V(f, a, c)

4. If f and $\alpha' \in R[a,b]$

Then

$$\int_a^b f(x)d\alpha(x)$$

is equal to

(a) $\int_a^b f(x)\alpha'(x)dx$ (b) $\int_a^b \alpha(x) f'(x)dx$

(c) $\int_a^b f(x)\alpha'(x)d\alpha$ (d) $\int_a^b \alpha(x) f'(x)d\alpha$

5. *Let* $f(x,y)=\sqrt{|xy|}$, *then* f *is*

(a) Continuous at (0,0) and differentiable at (0,0)
(b) Both continuous and differentiable at (0,0)
(c) Continuous at (0,0) but not differentiable at (0,0)
(d) None of the above

6. $\iint_S (x^3 dydz + y^3 dzdx + z^3 dxdy)$ *over the sphere* $x^2+y^2+z^2=a^2$ *is*

(a) $\frac{4}{3}\pi\ a^3$ (b) $4\pi\ a^2$

(c) $\frac{3}{4}\pi a^3$ (d) $\frac{12}{5}\pi a^5$

7. $A \subseteq R$ is measurable and x € R, then

$(a)\ \mu+(A+x)\neq\mu(A)$

$(b)\ \mu(A+x)>\mu(A)$

(c) $\mu(A+x)=\mu(A)$

(d) $\mu(A+x)<\mu(A)$

8. Measure of A = {2, 4, 6, 8,…} is
(a) ∞
(b) 2
(c) 0
(d) 1

9. If c is real and b is complex, the equation

$z\bar{z}+b\bar{z}+\bar{b}z+c=0$ represents
(a) straight line
(b) circle
(c) ellipse
(d) hyperbola

10. If f (z) is an analytic function with constant modulus, then f (z) is

(a) non-zero real
(b) complex
(c) zero
(d) positive real

11. If $f(z) = \frac{2z}{z+1}$ *and* $A = \{z : |z| = 1\}$

(a) straight line
(b) circle
(c) ellipse
(d) hyperbola

12. The value of

$$\int_0^{1+i} (x - y + ix^2)\,dz$$

along the straight line from z =0 and z =1+i is

$(a)\ \frac{1}{3}i$ $(b)\ \frac{1}{3}(1+i)$

$(c)\ \frac{1}{3}(i-1)$ $(d)\ \frac{1}{3}$

13. A simple pole of $\frac{1}{\sin z - \cos z}$ *is*

(a) 0 $(b)\frac{\pi}{2}$

$(c) -\frac{\pi}{4}$ $(d)\frac{\pi}{4}$

14. Residue of

$$\frac{z+1}{z^2(z-3)}$$

at z = 3 is

$(a)\frac{1}{9}$ $(b)\frac{2}{9}$

$(c)\frac{3}{9}$ $(d)\frac{4}{9}$

15. If

$$f(z)=\frac{z}{9-z^2}$$

$|z|=2$ and C is the circle

$$\int_C f(z)\frac{dz}{z+i} \quad \text{is}$$

$(a)\frac{\pi}{5}$ $\quad$ $(b)-\frac{\pi}{5}$

$(c)\frac{\pi i}{5}$ $\quad$ $(d)-\frac{\pi i}{5}$

16. Let

$$f(z)=\frac{1}{x^2+y^2}(x^3(1+i)-y^3(1-i))$$

If z ≠0 and f (0) =0, then at the origin f is

(a) continuous and $f'(0)$ exists
(b) continuous and *C*-*R* equations are satisfied
(c) discontinuous and C-R equations are satisfied
(d) discontinuous and $f'(0)$ does not exist

PART-B (Subjective-type Questions)
(Marks: 48)
Answer any **three** parts of each of the Question Nos. **17, 18, 19, 20**

17. (a) If $f \in R(\alpha)$

and k is a number such that $|f(x)|\le k$

for all x Є[a, b], then prove that

$$\left|\int_a^b f d\alpha\right| \le k\{\alpha(b)-\alpha(a)\}$$

(b) Prove that a monotonic increasing function which is bounded in closed interval [a,b] is a function of bounded variation. State its total variation.

(c) State and prove Weierstrass M-test of uniform convergence of a series of function.

(d) Show that the power series

$$1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\cdots$$

Is uniformly convergent on [-1, k], 0<k<1.

(c) Prove that every continuous function is measurable. 4×3=12

18. (a) If f and g are bounded measurable functions defined on a measurable set E of finite measure, then prove that

$$\int_E (\alpha f+\beta g)=\alpha\int_E f+\beta\int_E g$$

(b) If f is bounded and integrable on [-π, π] and if a_n, b_n are its Fourier coefficients, then prove that

$$\sum_{n\geq 1} a_n^2+b_n^2$$

is convergent

(c) Show that

$$\int_0^1 dx\int_0^1 \frac{x-y}{(x+y)^3}dy \neq \int_0^1 dy\int_0^1 \frac{x-y}{(x+y)^3}dx$$

(d) State and prove a theorem connecting a line integral along a closed contour with a double integral over the domain bounded by that contour.

(e) Compute

$$\iint_E \sqrt{a^2-x^2-y^2}\,dxdy$$

Where E is the region bounded by the circle $x^2+y^2= ax$. 4×3=12

19. (a) Find the analytical function f(x) = u + iv of which the real part is

$$u=e^x(x\cos y-y\sin y)$$

(b) Prove that the Cauchy-Riemann equations can be written in polar form as

(c) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ *and* $\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}$ Show that

$$\int_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3} dz = \frac{21\pi i}{16}$$

(d) Expand

$$f(z) = \frac{1}{(z+1)(z+3)}$$

In a Laurent series valid for $1 < |z| < 3$

(e) Show that 4×3=12

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$$

20. (a) State and prove Liouville's theorem.

(b) Prove that all the roots of

$$z^7 - 5z^3 + 12 = 0$$

lie between the circle $|z| = 1$ *and* $|z| = 2$

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(c) State Rouche's theorem and use it to show that every polynomial of degree n has exactly n zeros.

(d) If f: G → C is analytical, then prove that f preserves angles at each point z_0 of G, where $f'(z_0) \neq 0$.

(e) Find a bilinear transformation which maps the upper half of the z-plane into the unit circle in the ω = 0 and ∞ is mapped into ω = -1. 4×3=12